Athemath Spring 2024 Admissions Quiz

Athemath Staff

Due January 27th, 2024

§1 Instructions

For all problems, **proof-based solutions are encouraged**. We would like you to explain all of your steps, instead of just giving an answer. If you don't have experience with proofs, just try to explain your answer as much as you can. LaTeX submissions and *neat*, dark handwriting submissions are both allowed.

We also encourage you to try the entire test. While later problems will generally be harder, they also play to different strengths and you may find one particularly easy.

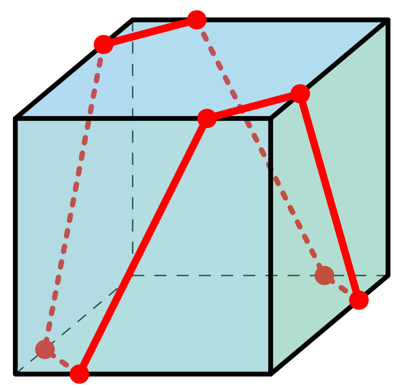
Please do not use computer programs, Google, WolframAlpha, etc. to help you find solutions (However, GeoGebra is allowed). Additionally, please do not discuss this quiz with anyone else until after the application deadline has passed. **If you find the test difficult, that's because it's designed to be.** If you get stuck, take a walk, try a different problem, or try a strategy you dismissed at first. And remember that you don't have to solve all of the problems to get in. Historically, the average admitted student solved around two or three problems.

Ask for clarifications by emailing Serena at serena.an@athemath.org. Submit your completed solutions to the application form by **January 27th, 11:59PM Eastern**. As a reminder, only students of underrepresented genders can apply. Have fun!

§ 2 The Problems!

Problem 1

A ribbon is wrapped around a cube of side length 1, as shown in the figure to the right, such that the ribbon cuts eight of the cube's edges in the same $x : 1-x$ ratio. Given that the length of ribbon on each of the six faces of the cube is the same, compute the total length of the ribbon.



Problem 2

Suppose A, B, and C are subsets of $\{1, 2, \ldots, 10\}$, each chosen independently and uniformly at random. What is the probability that $A \subseteq B \subseteq C$ (meaning A is a subset of B which is a subset of C)?

Problem 3

For each positive integer n, we define

$$f(n) = 1 \cdot 7^1 + 2 \cdot 7^2 + 3 \cdot 7^3 + \cdots + n \cdot 7^n.$$

For example, $f(4) = 1 \cdot 7 + 2 \cdot 7^2 + 3 \cdot 7^3 + 4 \cdot 7^4 = 7 + 98 + 1029 + 9604 = 10738$.

(a) Find the last two decimal digits of the number $f(20)$.

(b) Find the smallest positive integer n such that $f(n)$ ends in the digits 00.

Problem 4

In an equilateral triangular grid with n rows, where the i^{th} row has exactly i vertices, there are $n-1$ identical piggies stationed on the topmost vertex. In each row below the first row, exactly one vertex contains a *house*. Every second, each piggy not in a house moves to one of the two vertices adjacent to itself in the row below; if the vertex contains a house, the piggy enters it. If multiple piggies are able to move down to the same house, at most one may do so, and the remaining piggies must move to the other adjacent vertex. Determine the number of configurations of houses such that it is possible for all of the piggies to move into houses.

Problem 5

In a scalene triangle ABC with circumcenter O, let D be the foot of the altitude from A to BC. The line through D perpendicular to line AO intersects lines AB and AC at E and F, respectively. Suppose that the perpendicular bisector of EF intersects line AD at K. Prove that $KO \parallel BC$.

Problem 6

Vera and Celestia play a game. First, Celestia writes down an odd integer $n > 1$ on the board. On each turn, Vera replaces the number k currently on the board with either k^2 or $\frac{k+1}{2}$. Vera wins if she can write down Celestia's original number after a (nonzero) finite number of moves and every number written down in the process is an odd positive integer. Determine all starting numbers Celestia may write such that it is possible for Vera to win.
(An example of a sequence of moves that Vera may do if the starting number is 17 is $17 \to 9 \to 81 \to 41 \to 21 \to 11$. An example of a sequence of moves that causes Vera to win if the starting number is 3 is $3 \to 9 \to 5 \to 3$.)