Athemath Spring 2021 Admissions Quiz

Athemath Staff

Due February 27th, 2021

1 Instructions

For all the problems below, **proof-based answers are encouraged**. If you are not familiar with proofs, try to explain your thinking as best as you can. LaTeX submissions are encouraged, but *neat*, dark, cleanly-scanned handwriting is also permissible. **If your submission is not readable, we may not consider your application.**

We also encourage you to try the entire test. While later problems will generally be harder, they also play to different strengths and you may find one particularly easy. For multi-part problems, oftentimes the placement is solely determined by the difficulty of the last part, so you may find the first parts easier. **You may submit answers to later parts using results from previous parts even if you have not completed the previous parts.**

Please do not use calculators except on the problems where they are explicitly allowed. Additionally, please do not use computer programs, Google, WolframAlpha, GeoGebra, etc. to help you find solutions. In particular, please do not discuss this quiz with anyone else until after the application deadline has passed. If you need to cheat to get in, our courses probably aren't a good fit for you. **And if you find the test difficult, that's because it's designed to be.** Only hard problems are worth doing, after all! If you get stuck, take a walk, try small cases, listen to some music, try a different problem, or try a strategy you dismissed at first. And remember that you don't have to solve all—or even a majority—of the problems to get in.

Ask for clarifications by emailing Ali at president@athemath.org. Submit your completed solutions to the student application form by **February 27th 11:59PM Eastern**. As a reminder, only students of underrepresented genders can apply. Have fun!

2 The Problems!

Problem 1

Prair picks four integers a_1, a_2, a_3, and a_4 (not necessarily distinct) between 1 and 100, inclusive.

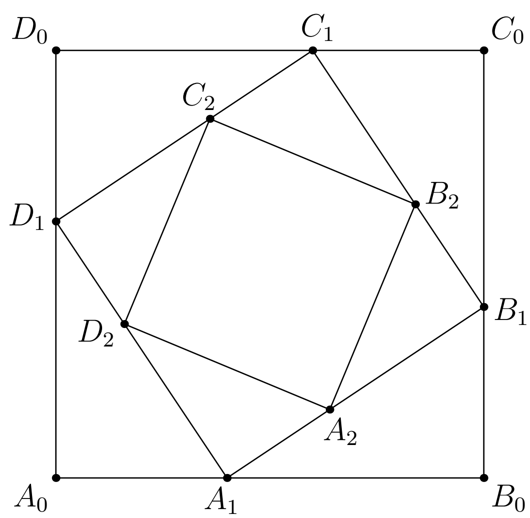
(a) Find the total number of ordered quadruplets (a_1, a_2, a_3, a_4) such that $a_1^2+a_2^2+a_3^2+a_4^2$ is divisible by 4. (You may leave your answer unsimplified.)

(b) Suppose instead that Prair picked five integers a_1, a_2, a_3, a_4, and a_5 (not necessarily distinct) between 1 and 100, inclusive. Find the total number of ordered quintuplets $(a_1, a_2, a_3, a_4, a_5)$ such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$ is divisible by 4. (You may leave your answer unsimplified.)

Problem 2

Let $A_0B_0C_0D_0$ be a square with side length 1, and suppose that r is a positive real number. Points A_i, B_i, C_i, and D_i, for $i \geq 1$, lie on segments $A_{i-1}B_{i-1}$, $B_{i-1}C_{i-1}$, $C_{i-1}D_{i-1}$, and $D_{i-1}A_{i-1}$, respectively, such that

$$\frac{A_iA_{i+1}}{A_{i+1}B_i} = \frac{B_iB_{i+1}}{B_{i+1}C_i} = \frac{C_iC_{i+1}}{C_{i+1}D_i} = \frac{D_iD_{i+1}}{D_{i+1}A_i} = r.$$



(a) If $[XYZ]$ denotes the area of triangle XYZ, find

$$[A_0A_1D_1] + [A_1A_2D_2] + [A_2A_3D_3] + \ldots + [A_jA_{j+1}D_{j+1}] + \ldots$$

(b) Suppose that the sides of $A_3B_3C_3D_3$ are parallel to those of $A_0B_0C_0D_0$.

(i) Find all possible values of r.

(ii) Find the area of $A_3B_3C_3D_3$.

(iii) Find an expression for the area of $A_nB_nC_nD_n$ in terms of n.

Problem 3

Let n be an integer. The sequence $\{a\} = a_1, a_2, \ldots, a_{4n}$ is a permutation of $1, 2, \ldots, 4n$, such that for any 4 indices $1 \leq i \leq j \leq k \leq l \leq 4n$, if $\dfrac{i+j+k+l}{4}$ is an integer, then $\dfrac{a_i + a_j + a_k + a_l}{4}$ is an integer.

(a) Show that if $i \equiv j \pmod 4$ then $a_i \equiv a_j \pmod 4$.

(b) Determine the number of possible sequences $\{a\}$ in terms of n.

Problem 4

You may use a calculator on this problem.

The *sigmoid function* S is a function on all real numbers used to compress the set of real numbers to the interval $(0, 1)$:

$$S(x) = \frac{1}{1 + e^{-x}}.$$

Define the set of *stretched sigmoid functions* to be the set of functions

$$S_\alpha(x) = \frac{1}{1 + \alpha^{-x}}$$

for real $\alpha > 1$.

(a) Explain why the range of $S(x) = \dfrac{1}{1 + e^{-x}}$ is $(0, 1)$ exactly.

(b) Find the range of $S_\alpha(x)$, an arbitrary stretched sigmoid function.

(c) Let function exponentiation denote repeated application; i.e., for any function f, $f^2(x) = f(f(x)$, $f^3(x) = f(f(f(x)))$, and in general,

$$f^n(x) = \underbrace{f(f(\cdots f(f(x))\cdots))}_{n \text{ times}}.$$

Find the range of $S_3^4(x)$.

(d) For which x does there exist $\alpha > 1$ such that x is in the range of S_α^n for all $n \geq 0$? If you can, prove your answer.